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Hawking Thermal Radiation of the Dirac Particle in Spherically Symmetric Nonstatic Space-Time

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The dynamical properties of Dirac spinor particles in a spherically symmetric nonstatic space-time are studied. The explicit representative of the four-component wave function of Dirac particles is obtained. The Dirac equation can be reduced to the standard form of the wave equation near the event horizon by the proper coordinate transformation. The event horizon location and Hawking radiation temperature are obtained.

1. INTRODUCTION

A significant development in quantum field theory over the past 20 years is the discovery of the quantum mechanical nonstability of the black hole (Gegenberg and Kunstatter, 1933). This important theoretical development not only solved a contradiction in black hole thermodynamics, but also deeply revealed the contact of quantum mechanics, thermodynamics, and gravity.

In the universe a black hole surely changes with time because of evaporation and accretion, so it is important to study its Hawking radiation in order to completely understand it.

A popular method to study a Hawking nonstatic black hole considering the radiation backreaction is to use the theory of quantum fields in curved space-time, but this theory is useful only for weak radiation and is complicated. In recent years, we suggested a new research approach and obtained the Hawking radiation temperature of a Klein–Gordon particle for every nonstatic space-time.

In this paper, we study the dynamic forms of the Dirac particle in spherically symmetric nonstatic space-time. By using the proper coordinate we obtain the solution of the equation of the Dirac particle, which shows

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that the static mass is not zero near the black hole event horizon in spherically symmetric nonstatic space-time, and we obtain the corresponding Hawking thermal spectrum formula.

2. THE DIRAC EQUATION IN SPHERICALLY SYMMETRIC NONSTATIC SPACE-TIME

In spherically symmetric nonstatic space-time, the linear element is (Zhao, 1993)

$$ds^2 = g_{00}dv^2 + 2g_{01}dv\,dr - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \tag{1}$$

We use the signature $(+, -, -, -)$, and $g_{00} = g_{00}(v, r)$, $g_{01} = g_{01}(v, r)$. The zero frames form of the metric is

$$\begin{aligned} ds^2 &= (l_\mu dx^\mu)(n_\nu dx^\nu) + (n_\mu dx^\mu)(l_\nu dx^\nu) \\ &\quad - (m_\mu dx^\mu)(\overline{m}_\nu dx^\nu) - (\overline{m}_\mu dx^\mu)(m_\nu dx^\nu) \end{aligned} \tag{2}$$

and we select the zero frames as follows

$$l_\mu = \left[\frac{1}{2}\, g_{00}, g_{01}, 0, 0\right], \qquad n_\mu = [1, 0.0, 0] \tag{3}$$

$$m_\mu = \frac{r}{\sqrt{2}}\,[0, 0.1, i \sin\theta], \qquad \overline{m}_\mu = \frac{r}{\sqrt{2}}\,[0.0, 1, - i \sin\theta]$$

From (3), we get

$$l^\mu = \left[1, -\frac{1}{2}\frac{g_{00}}{g_{01}}, 0, 0\right], \qquad n^\mu = \left[0, \frac{1}{g_{01}}, 0, 0\right] \tag{4}$$

$$m^\mu = \frac{1}{\sqrt{2}r}\left[0, 0, -1, -\frac{i}{\sin\theta}\right], \qquad \overline{m^\mu} = \frac{1}{\sqrt{2}}\left[0, 0, - 1, \frac{i}{\sin\theta}\right]$$

Equations (3) and (4) are zero vectors and satisfy pseudo-orthogonal and metric conditions. From (4) the direction differentials are

$$D = l^\mu \frac{\partial}{\partial x^\mu} = \frac{1}{g_{01}}\frac{\partial}{\partial r}$$

$$\delta = m^\mu \frac{\partial}{\partial x^\mu} = -\frac{1}{\sqrt{2}r}\left(\frac{\partial}{\partial\theta} + \frac{i\partial}{\sin\theta\,\partial\varphi}\right) \tag{5}$$

$$\overline{\delta} = \overline{m^\mu} \frac{\partial}{\partial x^\mu} = -\frac{1}{\sqrt{2}r}\left(\frac{\partial}{\partial\theta} - \frac{i\partial}{\sin\theta\,\partial\varphi}\right)$$

From (3) and (4), according to (4.1a) in Newman and Penrose (1963), we obtain the spin coefficients as follows:

$$\begin{aligned}
&\rho = \frac{g_{00}}{2rg_{01}}, \qquad \mu = \frac{1}{rg_{01}}, \qquad \alpha = \frac{1}{2\sqrt{2}r}\operatorname{ctg}\theta; \\
&\epsilon = \frac{1}{2g_{01}}\frac{\partial g_{01}}{\partial v} - \frac{1}{4g_{01}}\frac{\partial g_{00}}{\partial r} \\
&\beta = -\alpha, \qquad \kappa = \sigma = \tau = \gamma = \lambda = \pi = \nu = 0
\end{aligned} \tag{6}$$

The Dirac particle's field equation in spin coordinate form in curved space-time is

$$\nabla_{AB}P^A + i\frac{\mu_0}{\sqrt{2}}\overline{Q}_B = 0, \qquad \nabla_{AB}Q^A + i\frac{\mu_0}{\sqrt{2}}\overline{P}_B = 0 \tag{7}$$

where P^A and Q^A are two-component spinors, ∇_{AB} is the covariant differentiation, and μ_0 is the static mass of the Dirac particle.

Using the zero frames, (7) can be reduced to

$$\begin{aligned}
&(D + \epsilon - \rho)F_1 + (\overline{\delta} + \pi - \alpha)F_2 = i\frac{\mu_0}{\sqrt{2}}G_1 \\
&(\Delta + \mu - \gamma)F_2 + ((\delta + \beta - \tau)F_1 = i\frac{\mu_0}{\sqrt{2}}G^2 \\
&(D + \overline{\epsilon} - \overline{\rho})G_2 - (\delta + \overline{\pi} - \overline{\alpha})G_1 = i\frac{\mu_0}{\sqrt{2}}F_2 \\
&(\Delta + \overline{\mu} - \overline{\gamma})G_1 - (\overline{\delta} + \overline{\beta} - \overline{\tau})G_2 = i\frac{\mu_0}{\sqrt{2}}F_1
\end{aligned} \tag{8}$$

where $F_1 = P^0$, $F_2 = P^1$, $G_1 = \overline{Q^1}$, and $G_2 = -\overline{Q^0}$. The explicit representations of D, Δ, δ and ϵ, ρ, π, α, β, τ, μ, and γ are (5) and (6), respectively.

Substituting (5) and (6) into (8), we get

$$\begin{aligned}
&\left(\frac{\partial}{\partial v} - \frac{g_{00}}{2g_{01}}\frac{\partial}{\partial r} + \frac{1}{2g_{01}}\frac{\partial g_{01}}{\partial v} - \frac{1}{4g_{01}}\frac{\partial g_{00}}{\partial r} - \frac{g_{00}}{2rg_{01}}\right)f_1 \\
&\quad - \frac{1}{\sqrt{2}r}\left(\frac{\partial}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial}{\partial\varphi} + \frac{1}{2}\operatorname{ctg}\theta\right)F_2 - \frac{i}{\sqrt{2}}\mu_0 G_1 = 0 \\
&\left(\frac{1}{g_{01}}\frac{\partial}{\partial r} + \frac{1}{rg_{01}}\right)f_2 - \frac{1}{\sqrt{2}r}\left(\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\varphi} + \frac{1}{2}\operatorname{ctg}\theta\right)F_1 - \frac{i}{\sqrt{2}}\mu_0 G_2 = 0
\end{aligned} \tag{9}$$

$$\left(\frac{\partial}{\partial v} - \frac{g_{00}}{2g_{01}}\frac{\partial}{\partial r} + \frac{1}{2g_{01}}\frac{\partial g_{01}}{\partial v} - \frac{1}{4g_{01}}\frac{\partial g_{00}}{\partial r} - \frac{g_{00}}{2rg_{01}}\right)g_2$$

$$+ \frac{1}{\sqrt{2}r}\left(\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\varphi} + \frac{1}{2}\operatorname{ctg}\theta\right)G_1 - \frac{i}{\sqrt{2}}\mu_0 F_2 = 0$$

$$\left(\frac{1}{g_{01}}\frac{\partial}{\partial r} + \frac{1}{rg_{01}}\right)g_1 + \frac{1}{\sqrt{2}r}\left(\frac{\partial}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial}{\partial\varphi} + \frac{1}{2}\operatorname{ctg}\theta\right)G_2 - \frac{i}{\sqrt{2}}\mu_0 F_1 = 0$$

Assuming that

$$F_1 = \epsilon^{im\varphi}f_1(v, r, \theta), \qquad F_2 = \epsilon^{im\varphi}f_2(v, r, \theta)$$
$$G_1 = e^{im\varphi}g_1(v, r, \theta), \qquad G_2 = \epsilon^{im\varphi}g_2(v, r, \theta) \tag{10}$$

we separate Variables as follows:

$$f_1 = R_-(v, r)S_-(\theta), \qquad f_2 = R_+(v, r)S_+(\theta)$$
$$g_1 = R_-(v, r)S_-(\theta), \qquad g_2 = R_-(v, r)S_+(\theta) \tag{11}$$

Thus (9) can be reduced to

$$\frac{\sqrt{2}}{g_{01}}ER_- = (i\mu_0\tau + \lambda)R_+ \tag{12}$$

$$\frac{\sqrt{2}}{g_{01}}E^+R_+ = (i\mu_0 r - \lambda)R_- \tag{13}$$

$$F^+S_+ = \lambda S_- \tag{14}$$

$$FS_- = -\lambda S_+ \tag{15}$$

where

$$E = g_{01}r\frac{\partial}{\partial r} - \frac{r}{2}g_{00}\frac{\partial}{\partial r} + \frac{r}{2}\frac{\partial g_{01}}{\partial v} - \frac{r}{4}\frac{\partial g_{00}}{\partial r} - \frac{1}{2}g_{00}, \qquad E^+ = r\frac{\partial}{\partial r} + 1$$

$$F = \frac{\partial}{\partial\theta} - \frac{m}{\sin\theta} + \frac{1}{2}\operatorname{ctg}\theta, \qquad F^+ = \frac{\partial}{\partial\theta} + \frac{m}{\sin\theta} + \frac{1}{2}\operatorname{ctg}\theta$$

λ is a constant of separation of variables. The radial equation can be reduced to

$$\frac{g_{00}}{g_{01}}\frac{\partial^2 R}{\partial r^2} - 2\frac{\partial^2 R_-}{\partial r\,\partial v} + \left(\frac{2i\mu_0}{i\mu_0 r + \lambda} - \frac{4}{r}\right)\frac{\partial R_-}{\partial v} + \left(\frac{3g_{00}}{rg_{01}} - \frac{1}{g_{01}}\frac{\partial g_{01}}{\partial v}\right.$$
$$\left. + \frac{3}{2g_{01}}\frac{\partial g_{01}}{\partial r} - \frac{g_{00}}{g_{01}^2}\frac{\partial g_{01}}{\partial r} - \frac{i\mu_0 r g_{00}}{(ir\mu_0 + \lambda)g_{01}}\right)\frac{\partial r_-}{\partial r} + B_-R_- = 0 \tag{16}$$

where

$$B_- = \frac{1}{g_{01}^2}\frac{\partial g_{01}}{\partial r}\frac{\partial g_{01}}{\partial v} - \frac{1}{g_{01}}\frac{\partial^2 g_{01}}{\partial r \partial v} + \frac{1}{2g_{01}}\frac{\partial^2 g_{00}}{\partial r^2} - \frac{1}{2g_{01}^2}\frac{\partial g_{01}}{\partial r}\frac{\partial g_{00}}{\partial r} + \frac{1}{rg_{01}}\frac{\partial g_{00}}{\partial r}$$

$$- \frac{g_{00}}{rg_{01}^2}\frac{\partial g_{01}}{\partial r} + \frac{g_{00}}{r^2 g_{01}} - \frac{1}{rg_{01}}\frac{\partial g_{01}}{\partial v} - \frac{1}{r^2}(\lambda^2 + \mu_0^2 r^2)g_{01}$$

$$+ \frac{i\mu_0}{\lambda + i\mu_0 r}\left(\frac{1}{g_{01}}\frac{\partial g_{01}}{\partial v} - \frac{1}{2g_{01}}\frac{\partial g_{00}}{\partial r} - \frac{g_{00}}{rg_{01}}\right)$$

3. HAWKING EVAPORATION OF DIRAC PARTICLE NEAR THE BLACK HOLE HORIZON

Introducing the tortoise coordinates (Zhao *et al.*, 1996)

$$r_* = \ln[r - r_H(v)], \qquad v_* = \int A(\tau_H, v)\, dv \tag{17}$$

where $r_H(v)$ is the horizon radius of the black hole, we have

$$A(r_H, v) = -\left[\frac{g_{00} + 2\dot{r}_H g'_{01}}{g_{01}} + \frac{G}{2\omega}i\right]_{r=rH}$$

$$G + \left[\frac{1}{g_{01}}\frac{\partial g_{01}}{\partial v} - \frac{1}{2g_{01}}\frac{\partial g_{01}}{\partial r} - \frac{g_{00}}{r_H g_{01}}\right]_{r=rH}$$

Here ω is a constant

Equation (16) can be reduced to

$$\frac{g_{00}/g_{01} + \dot{r}_H}{r - r_H}\frac{\partial^2 R_-}{\partial r_*^2} - 2A\frac{\partial^2 R_-}{\partial v \partial r_*} - \left[\frac{g_{00}}{g_{01}(r - r_H)} + \frac{2\dot{r}_H}{r - r_H}\right.$$

$$\left. + \dot{r}_H\left(\frac{i2\mu_0}{i\mu_0 r + \lambda} - \frac{4}{r}\right) - \frac{3}{2g_{01}}\frac{\partial g_{01}}{\partial r} + \frac{g_{00}}{g_{01}^2}\frac{\partial g_{01}}{\partial r} + \frac{i\mu_0 r g_{00}}{(ir\mu_0 + \lambda)g_{01}}\right]\frac{\partial R_-}{\partial r_*}$$

$$+ (r - r_H)B_- R_- + \left(\frac{i2\mu_0}{i\mu_0 r + \lambda} - \frac{4}{r}\right)a(r - r_H)\frac{\partial R_-}{\partial v_*} = 0 \tag{18}$$

We will consider the asympotic forms of (18) near the horizon when $r - r_H$. In order to make the limit of the coefficient of $\partial^2 R-/\partial r_*^2$ finite, we must demand that the particle satisfy

$$\lim_{r \to r_H} [g_{00} + 2\dot{r}_H g_{01}] = 0 \tag{19}$$

That is,

$$[g_{00} + 2\dot{r}_H g_{01}]_{r=r_H} = 0 \tag{20}$$

Equation (20) is the equation that decides the location of black hole horizon surface; it is in concordance with the conclusion of the null-surface equation.

Thus

$$\lim_{r-r_H} \frac{g_{00} + 2\dot{r}_H g_{01}}{g_{01}(r - r_H)} = \left.\frac{g'_{00} + 2\dot{r}_H g'_{01}}{g_{01}}\right|_{r=r_H} \tag{21}$$

where $g'_{00} = \partial g_{00}/\partial r$, $g'_{01} = \partial g_{01}/\partial r$. Then, near $r = r_H$; (18) can be reduced to

$$-\left[\frac{g'_{00} + 2\dot{r}_H g'_{01}}{g_{01}}\right]_{r=r_H} \frac{\partial^2 R_-}{\partial r_*^2} + 2A \frac{\partial^2 R_-}{\partial v_* \partial r_*} + G \frac{\partial R_-}{\partial r_*} = 0 \tag{22}$$

Two solutions for (22) are

$$R_-^{\text{in}} = e^{-i\omega v_*} \tag{23}$$

$$R_-^{\text{out}} = e^{-i\omega v_*} e^{2i\omega r_*} \tag{24}$$

We know the vibration frequency in (23) and (24) at the condition that the time coordinate is v_*; now we let $\overline{w}$ be the vibration frequency when the time coordinate is v; thus

$$\omega v_* = \overline{\omega} v \tag{25}$$

From (17) we get

$$\omega = \frac{\overline{\omega}}{\kappa} + \frac{i\omega'}{\kappa} \tag{26}$$

where

$$\omega' = \frac{1}{2v} \int G\, dv \qquad \kappa = \frac{-1}{v} \int \left[\frac{g'_{00} + 2\dot{r}_H g'_{01}}{g_{01}}\right]_{r=r_H} dv$$

Then (23) and (29) can be reduced to

$$R_-^{\text{in}} = e^{-i\overline{\omega} r} \tag{27}$$

$$R_-^{\text{out}} = e^{-\overline{\omega} r} e^{2i\overline{\omega} r_* i/\kappa} e^{-2\omega' r_*/\kappa} \tag{28}$$

(28) is not analytic at $r = r_H$; according to Damour and Ruffini (1976), the only analytic continuation to the inside of the horizon through the lower half-plane,

$$(r - r_H) - |r - r_H| \epsilon^{-i\pi} = (r_H - r)\epsilon^{-i\pi} \tag{29}$$

Thus

$$R_{-}^{\text{out}} = \begin{cases} e^{-i\bar{\omega}v} e^{2i\bar{\omega}r_*/\kappa} e^{-2\omega' r_*/\kappa}, & r > r_H \\ R_{-}^{\text{out}} e^{2\bar{\omega}/\kappa} e^{2\omega' i/\kappa}, & r < r_H \end{cases} \tag{30}$$

The scattering probability of the outgoing wave R^{out} at the horizon is

$$\left|\frac{R_{-}^{\text{out}}}{\bar{R}_{-}^{\text{out}}}\right|^2 = e^{-4\pi\bar{\omega}i/\kappa} \tag{31}$$

where

$$\bar{R}_{-}^{\text{out}} = R_{-}^{\text{out}} \epsilon^{2\bar{\omega}/\kappa} \epsilon^{2\omega' i/\kappa}$$

Following Sannan (1988) we have

$$N_{\bar{\omega}} = \frac{1}{\epsilon^{\bar{\omega}/T} + 1} \tag{32}$$

$$\Gamma = \frac{-1}{4\pi v} \int \left[\frac{g'_{00} + 2\dot{r}_H g'_{01}}{g_{01}}\right]_{r=r_H} dv \tag{33}$$

4. CONCLUSION

Equation (32) is the Hawking radiation thermal spectrum formula of a Dirac particle on a spherically symmetric nonstatic black hole horizon surface. It contains a temperature parameter determined by the space-time metric components g_{00} and g_{01}.

There are some particular examples as follows:

1. Vaidya black hole (Balbinot, 1986; York, 1984):

$$ds^2 = \left[1 - \frac{2M(v)}{r}\right] dv^2 - 2dv\, dr - r^2\, d\Omega^2 \tag{34}$$

Substituting (34) into (20) and (33), we get

$$r_H = \frac{2M(v)}{1 - 2\dot{r}_H} \tag{35}$$

$$T = \frac{1}{2\pi v} \int \frac{M(v)}{r_H^2}\, dv \tag{36}$$

2. The spherically symmetric black hole (York, 1984)

$$ds^2 = e^{2\psi}\left(1 - \frac{2M(v)}{r}\right) dv^2 - 2e^{\psi}\, dv\, dr - r^2\, d\Omega^2 \tag{37}$$

where $\psi = \psi(v, r)$, $M = M(v, r)$.

Substituting (37) into (20) and (33), we get

$$r_H = \frac{2M(v)}{1 - 2r_H e^{-\psi}} \tag{38}$$

$$T = \frac{1}{4\pi i} \int \left[e^{\psi}\left(1 - \frac{2M}{r}\right) \frac{\partial \psi}{\partial r} + 2e^{\psi} \frac{\partial M}{\partial r} + e^{\psi} \frac{2M}{r} - 2\dot{r}_H \frac{\partial \psi}{\partial r} \right]_{r=r_H} dv \tag{39}$$

From the discussion above, the energy spectrum of the Dirac particle near a black hole horizon surface has the form of black hole radiation. We obtain the equations giving the location of the event horizon and the Hawking radiation temperature. These 4-results coincide with the results from studying scalar particles (York, 1984). We avoid the difficulty of finding the energy-momeutum tensor and provide a way for studying the thermal effect of Dirac particles in nonstatic space-time.

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